

Reconstruction of scalar potentials in two-field cosmological models

Alexander A. Andrianov

V.A. Fock Department of Theoretical Physics, Saint Petersburg State University,
198904, S.-Petersburg, Russia
Departament d'Estructura i Constituents de la Materia, Universitat de Barcelona,
08028, Barcelona, Spain

Francesco Cannata

INFN, Via Irnerio 46, 40126 Bologna, Italy

Alexander Y. Kamenshchik

Dipartimento di Fisica and INFN, Via Irnerio 46, 40126 Bologna, Italy
L.D. Landau Institute for Theoretical Physics of the Russian Academy of Sciences,
Kosygin str. 2, 119334 Moscow, Russia

Daniele Regoli

Dipartimento di Fisica, Via Irnerio 46, 40126 Bologna, Italy

E-mail: andrianov@bo.infn.it, cannata@bo.infn.it, kamenshchik@bo.infn.it
and astapovo@gmail.com

Abstract. We study the procedure of the reconstruction of phantom-scalar field potentials in two-field cosmological models. It is shown that while in the one-field case the chosen cosmological evolution defines uniquely the form of the scalar potential, in the two-field case one has an infinite number of possibilities. The classification of a large class of possible potentials is presented and the dependence of cosmological dynamics on the choice of initial conditions is investigated qualitatively and numerically for two particular models.

PACS numbers: 98.80.Cq, 98.80.Jk

Keywords: phantom dark energy, scalar fields

1. Introduction

The discovery of cosmic acceleration [1] has stimulated the construction of a class of dark energy models [2, 3, 4] describing this effect. Notice that the cosmological models based on scalar fields were considered long before the observational discovery of cosmic acceleration [5]. The dark energy should possess a negative pressure such that the relation between pressure and energy density w is less than $-1/3$. The present day value of the parameter w is close to -1 (cosmological constant) but some observations indicate that the value of this parameter slightly less than -1 provides the best fit. The corresponding dark energy has been named phantom dark energy [6]. According to some authors, the analysis of observations permits the existence of the moment when the universe changes the value of the parameter w from $w > -1$ to $w < -1$ [7, 8]. This transition is called “the crossing of the phantom divide line”. The most recent investigations have shown that the phantom divide line crossing is still not excluded by the data [9].

It is easy to see, that the standard minimally coupled scalar field cannot give rise to the phantom dark energy, because in this model the absolute value of energy density is always greater than that of pressure, i.e. $|w| < 1$. A possible way out of this situation is the consideration of the scalar field models with the negative kinetic term. Thus, the important problem arising in connection with the phantom energy is the crossing of the phantom divide line. The general belief is that while this crossing is not admissible in simple minimally coupled models its explanation requires more complicated models such as “multifield” ones or models with non-minimal coupling between scalar field and gravity (see e.g. [10]).

In preceding papers some of us [11, 12] described the phenomenon of the change of sign of the kinetic term of the scalar field implied by the Einstein equations. It was shown that such a change is possible only when the potential of the scalar field possesses some cusps and, moreover, for some very special initial conditions on the time derivatives and values of the considered scalar field approaching to the phantom divide line. At the same time, two-field models including one standard scalar field and a phantom field can describe the phenomenon of the (de)-phantomization under very general conditions and using rather simple potentials [13, 14]. In the present paper we would like to attract attention to a drastic difference between two- and one-field models.

The reconstruction procedure of the scalar field potential models is well-known [15, 16, 17, 18, 19, 20, 21, 22]. We recapitulate it briefly. The cosmological evolution in a flat Friedmann model with the metric

$$ds^2 = dt^2 - a^2(t)dl^2 \quad (1)$$

can be given by the time evolution of the Hubble parameter $h(t) \equiv \frac{\dot{a}}{a}$, which satisfies the Friedmann equation

$$h^2 = \varepsilon, \quad (2)$$

where ε is the energy density and a convenient normalization of the Newton constant is chosen. Differentiating equation (2) and using the energy conservation equation

$$\dot{\varepsilon} = -3h(\varepsilon + p), \quad (3)$$

where p is the pressure, one comes to

$$\dot{h} = -\frac{3}{2}(\varepsilon + p). \quad (4)$$

If the matter is represented by a spatially homogeneous minimally coupled scalar field, then the energy density and the pressure are given by the formulae

$$\varepsilon = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad (5)$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (6)$$

where $V(\phi)$ is a scalar field potential. Combining equations (2), (4), (5), (6) we have

$$V = \frac{\dot{h}}{3} + h^2, \quad (7)$$

and

$$\dot{\phi}^2 = -\frac{2}{3}\dot{h}. \quad (8)$$

Equation (7) represents the potential as a function of time t . Integrating equation (8) one can find the scalar field as a function of time. Inverting this dependence we can obtain the time parameter as a function of ϕ and substituting the corresponding formula into equation (7) one arrives to the uniquely reconstructed potential $V(\phi)$. It is necessary to stress that this potential reproduces a given cosmological evolution only for some special choice of initial conditions on the scalar field and its time derivative [17, 11]. Below we shall show that in the case of two scalar fields one has an enormous freedom in the choice of the two-field potential providing the same cosmological evolution. This freedom is connected with the fact that the kinetic term has now two contributions. In order to examine the problem of phantom divide line crossing we shall be interested here in the case of one standard scalar field and one phantom field ξ , whose kinetic term has a negative sign. In this case the total energy density and pressure will be given by

$$\varepsilon = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\xi}^2 + V(\phi, \xi), \quad (9)$$

$$p = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\xi}^2 - V(\phi, \xi). \quad (10)$$

The relation (7) expressing the potential as a function of t does not change in form, but instead of equation (8) we have

$$\dot{\phi}^2 - \dot{\xi}^2 = -\frac{2}{3}\dot{h}. \quad (11)$$

Now, one has rather a wide freedom in the choice of the time dependence of one of two fields. After that the time dependence of the second field can be found from equation

(11). However, the freedom is not yet exhausted. Indeed, having two representations for the time parameter t as a function of ϕ or ξ , one can construct an infinite number of potentials $V(\phi, \xi)$ using the formula (7) and some rather loose consistency conditions. It is rather difficult to characterize all the family of possible two-field potentials, reproducing given cosmological evolution $h(t)$. In the present paper, we describe some general principles of construction of such potentials, then we consider some concrete examples.

2. The system of equations for two-field cosmological model

The system of equations, which we study contains (7) and (11) and two Klein-Gordon equations

$$\ddot{\phi} + 3h\dot{\phi} + \frac{\partial V(\phi, \xi)}{\partial \phi} = 0, \quad (12)$$

$$\ddot{\xi} + 3h\dot{\xi} - \frac{\partial V(\phi, \xi)}{\partial \xi} = 0. \quad (13)$$

From equations (12) and (13) we can find the partial derivatives $\frac{\partial V(\phi, \xi)}{\partial \phi}$ and $\frac{\partial V(\phi, \xi)}{\partial \xi}$ as functions of time t . The consistency relation

$$\dot{V} = \frac{\partial V(\phi, \xi)}{\partial \phi}(t)\dot{\phi} + \frac{\partial V(\phi, \xi)}{\partial \xi}(t)\dot{\xi} \quad (14)$$

is respected.

Before starting the construction of potentials for particular cosmological evolutions, it is useful to consider some mathematical aspects of the problem of reconstruction of a function of two variables in general terms.

2.1. Reconstruction of the function of two variables, which in turn depends on a third parameter

Let us consider the function of two variables $F(x, y)$ defined on a curve, parameterized by t . Suppose that we know the function $F(t)$ and its partial derivatives as functions of t :

$$F(x(t), y(t)) = F(t), \quad (15)$$

$$\frac{\partial F(x(t), y(t))}{\partial x} = \frac{\partial F}{\partial x}(t), \quad (16)$$

$$\frac{\partial F(x(t), y(t))}{\partial y} = \frac{\partial F}{\partial y}(t). \quad (17)$$

These three functions should satisfy the consistency relation

$$\dot{F}(t) = \frac{\partial F}{\partial x}(t)\dot{x} + \frac{\partial F}{\partial y}(t)\dot{y}. \quad (18)$$

As a simple example we can consider the curve

$$x(t) = t, \quad y(t) = t^2, \quad (19)$$

while

$$F(t) = t^2, \quad (20)$$

$$\frac{\partial F}{\partial x} = t^2, \quad (21)$$

$$\frac{\partial F}{\partial y} = t, \quad (22)$$

and equation (18) is satisfied.

Thus, we would like to reconstruct the function $F(x, y)$ having explicit expressions in right-hand side of equations (15)–(17). This reconstruction is not unique. We shall begin the reconstruction process taking such simple ansatzes as

$$F_1(x, y) = G_1(x) + H_1(y), \quad (23)$$

$$F_2(x, y) = G_2(x)H_2(y), \quad (24)$$

$$F_3(x, y) = (G_3(x) + H_3(y))^\alpha. \quad (25)$$

The assumption (23) immediately implies

$$\frac{\partial F_1}{\partial x} = \frac{\partial G_1}{\partial x}, \quad (26)$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial H_1}{\partial y}. \quad (27)$$

Therefrom one obtains

$$G_1(x) = \int^x \frac{\partial F_1}{\partial x'}(t(x'))dx', \quad (28)$$

$$H_1(y) = \int^y \frac{\partial H_1}{\partial y'}(t(y'))dy'. \quad (29)$$

Hence

$$F_1(x, y) = \int^x \frac{\partial F_1}{\partial x'}(t(x'))dx' + \int^y \frac{\partial H_1}{\partial y'}(t(y'))dy'. \quad (30)$$

For an example given by equations (19)–(22) the function $F_1(x, y)$ is

$$F_1(x, y) = \int^x t^2(x')dx' + \int^y t(y')dy' = \int^x x'^2 dx' \pm \int^y \sqrt{y'} dy'. \quad (31)$$

Explicitly

$$F_1(x, y) = \begin{cases} \frac{1}{3} (x^3 + 2y^{3/2}), & x > 0, \\ \frac{1}{3} (x^3 - 2y^{3/2}), & x < 0. \end{cases} \quad (32)$$

Similar reasonings give for the assumptions (24), and (25) correspondingly

$$F_2(x, y) = \exp \left\{ \int \left(\frac{1}{F} \frac{\partial F}{\partial x} \right) (t(x)) dx + \int \left(\frac{1}{F} \frac{\partial F}{\partial y} \right) (t(y)) dy \right\}, \quad (33)$$

$$F_3(x, y) = \left\{ \int \frac{dx}{\alpha} \left(F^{\frac{1}{\alpha}-1} \frac{\partial F}{\partial x} \right) (t(x)) + \int \frac{dy}{\alpha} \left(F^{\frac{1}{\alpha}-1} \frac{\partial F}{\partial y} \right) (t(y)) \right\}^\alpha. \quad (34)$$

For our simple example (19)–(22) the functions $F_2(x, y)$ and $F_3(x, y)$ have the form

$$F_2(x, y) = xy, \quad (35)$$

$$F_3(x, y) = \begin{cases} \left[\frac{1}{3} (x^{3/\alpha} + 2y^{3/2\alpha}) \right]^\alpha, & x > 0, \\ \left[\frac{1}{3} (x^{3/\alpha} (-)^{3/\alpha} 2y^{3/2\alpha}) \right]^\alpha, & x < 0. \end{cases} \quad (36)$$

Thus, we have seen that the same input of “time” functions (20)–(22) on the curve (19) produces quite different functions of variables x and y .

Naturally, one can introduce many other assumptions for reconstruction of $F(x, y)$. For example, one can consider linear combinations of x and y as functions of the parameter t and decompose the presumed function F as a sum or a product of the functions of these new variables.

Now we present a way of constructing the whole family of solutions starting from a given one. Let us suppose that we have a function $F_0(x, y)$ satisfying all the necessary conditions. Let us take an arbitrary function

$$f(x, y) = f \left(\frac{t(x)}{t(y)} \right), \quad (37)$$

which depends only on the ratio $t(x)/t(y)$. We require also

$$f(1) = 1, \quad f'(1) = 0, \quad (38)$$

i.e. the function reduces to unity and its derivative vanishes on the curve $(x(t), y(t))$. Then it is obvious that the function

$$F(x, y) = F_0(x, y) f \left(\frac{t(x)}{t(y)} \right) \quad (39)$$

is also a solution. This permits us to generate a whole family of solutions, depending on a choice of the function f . Moreover, one can construct other solutions, adding to the function $F(x, y)$ a term proportional to $(t(x) - t(y))^2$.

2.2. Cosmological applications, an evolution “Bang to Rip”

To show how this procedure works in cosmology, we consider a relatively simple cosmological evolution, which nevertheless is of particular interest, because it describes the phantom divide line crossing. Let us suppose that the Hubble variable for this evolution behaves as

$$h(t) = \frac{A}{t(t_R - t)}, \quad (40)$$

where A is a positive constant. At the beginning of the cosmological evolution, when $t \rightarrow 0$ the universe is born from the standard Big Bang type cosmological singularity, because $h(t) \sim 1/t$. Then, when $t \rightarrow t_R$, the universe is superaccelerated, approaching the Big Rip singularity $h(t) \sim 1/(t_R - t)$. Substituting the function (40) and its time derivative into equations. (11) and (7) we come to

$$\dot{\phi}^2 - \dot{\xi}^2 = -\frac{2A(2t - t_R)}{3t^2(t_R - t)^2}, \quad (41)$$

$$V(t) = \frac{A(2t - t_R + 3A)}{3t^2(t_R - t)^2}. \quad (42)$$

For convenience we choose also the parameter A as

$$A = \frac{t_R}{3}, \quad (43)$$

Then,

$$h(t) = \frac{t_R}{3t(t_R - t)}, \quad (44)$$

and

$$V(t) = \frac{2t_R}{9t(t_R - t)^2}. \quad (45)$$

Let us consider now a special choice of functions $\phi(t)$ and $\xi(t)$ used already[‡] in [13],

$$\phi(t) = -\frac{4}{3} \operatorname{arctanh} \sqrt{\frac{t_R - t}{t_R}}, \quad (46)$$

$$\xi(t) = \frac{\sqrt{2}}{3} \ln \frac{t}{t_R - t}. \quad (47)$$

The derivatives of the potential with respect to the fields ϕ and ξ could be found from the Klein-Gordon equations (12) and (13):

$$\frac{\partial V}{\partial \xi} = \ddot{\xi} + 3h\dot{\xi} = \frac{\sqrt{2}}{3} \frac{2t_R}{t(t_R - t)^2}, \quad (48)$$

$$\frac{\partial V}{\partial \phi} = -\ddot{\phi} - 3h\dot{\phi} = -\frac{\sqrt{t_R}}{t(t_R - t)^{3/2}}. \quad (49)$$

We can obtain also the time parameter as a function of ϕ or ξ :

$$t(\phi) = \frac{t_R}{\cosh^2(-3\phi/4)}, \quad (50)$$

$$t(\xi) = \frac{t_R}{\exp(-3\xi/\sqrt{2}) + 1}. \quad (51)$$

Now we can make a hypothesis about the structure of the potential $V(\xi, \phi)$:

$$V_2(\xi, \phi) = G(\xi)H(\phi). \quad (52)$$

[‡] Notice that the origin of two scalar fields has been associated in [13] with a non-Hermitian complex scalar field theory and there a classical solution was found as a saddle point in "double" complexification.

Applying the technique described in the subsection A, we can get $G(\xi)$:

$$\begin{aligned}\ln G(\xi) &= \int \left(\frac{1}{V} \frac{\partial V}{\partial \xi} \right) (t(\xi)) d\xi \\ &= \int \frac{9t(\xi)(t(\xi) - t_R)^2}{2t_R} \frac{\sqrt{2}}{3} \frac{2t_R}{t(\xi)(t_R - t(\xi))^2} d\xi = 3\sqrt{2}\xi,\end{aligned}\quad (53)$$

and

$$G(\xi) = \exp(3\sqrt{2}\xi). \quad (54)$$

To find $H(\phi)$ one can use the analogous direct integration, but we prefer to implement a formula

$$H(\phi) = \frac{V(t(\phi))}{G(\xi(t(\phi)))}, \quad (55)$$

which gives

$$H(\phi(t)) = \frac{2t_R}{9t^3}, \quad (56)$$

and hence,

$$H(\phi) = \frac{2}{9t_R^2} \cosh^6(-3\phi/4). \quad (57)$$

Finally,

$$V_2(\xi, \phi) = \frac{2}{9t_R^2} \cosh^6(-3\phi/4) \exp(3\sqrt{2}\xi). \quad (58)$$

Here we have reproduced the potential studied in [13].

Making the choice

$$V_1(\xi, \phi) = G(\xi) + H(\phi), \quad (59)$$

we derive

$$\begin{aligned}V_1(\xi, \phi) &= \frac{2}{3t_R^2} \left[-\frac{1}{3}e^{-3\xi/\sqrt{2}} + 3\sqrt{2}\xi + 2e^{3\xi/\sqrt{2}} + \frac{1}{3}e^{6\xi/\sqrt{2}} \right. \\ &\quad \left. + \frac{\sinh^4(-3\phi/4) + \sinh^2(-3\phi/4) - 1}{\sinh^2(-3\phi/4)} + \ln \sinh^4(-3\phi/4) \right].\end{aligned}\quad (60)$$

Now, we can make another choice of the field functions $\phi(t)$ and $\xi(t)$, satisfying the condition (41):

$$\phi(t) = \frac{\sqrt{2}}{3} \ln \frac{t}{t_R - t}, \quad (61)$$

$$\xi(t) = \frac{4}{3} \operatorname{arctanh} \sqrt{\frac{t}{t_R}}. \quad (62)$$

The time parameter t is a function of fields is

$$t(\phi) = \frac{t_R}{\exp(-3\phi/\sqrt{2}) + 1}, \quad (63)$$

$$t(\xi) = \frac{t_R}{\tanh^2(3\xi/4)}. \quad (64)$$

Looking for the potential as a sum of functions of two fields as in equation (59) after lengthy but straightforward calculations we come to the following potential:

$$\begin{aligned} V_1(\xi, \phi) = \frac{2}{3t_R^2} \left[1 + \frac{2}{3} \sinh^4(3\xi/4) + 3 \sinh^2(3\xi/4) - \frac{1}{3 \sinh^2(3\xi/4)} \right. \\ \left. + 2 \ln \sinh^2(3\xi/4) + \frac{2}{3} \exp(-3\phi/\sqrt{2}) - 3\sqrt{2}\phi \right. \\ \left. - 2 \exp(3\phi/\sqrt{2}) - \frac{1}{3} \exp(\phi/\sqrt{2}) \right]. \end{aligned} \quad (65)$$

Similarly for the potential designed as a product of functions of two fields (52) we obtain

$$V_2(\xi, \phi) = \frac{2}{9t_R^2} \sinh^2(3\xi/4) \cosh^2(3\xi/4) \exp(-3\sqrt{2}\phi). \quad (66)$$

One can make also other choices of functions $\phi(t)$ and $\xi(t)$ generating other potentials, but we shall not do it here (see [23]), concentrating instead on the qualitative and numerical analysis of two toy cosmological models described by potentials (65) and (58).

3. Analysis of cosmological models

It is well known [24] that for the qualitative analysis of the system of cosmological equations it is convenient to present it as a dynamical system, i.e. a system of first-order differential equations. Introducing the new variables x and y we can write

$$\begin{cases} \dot{\phi} = x, \\ \dot{\xi} = y, \\ \dot{x} = -3\text{sign}(h)x\sqrt{\frac{x^2}{2} - \frac{y^2}{2} + V(\xi, \phi)} - \frac{\partial V}{\partial \phi}, \\ \dot{y} = -3\text{sign}(h)y\sqrt{\frac{x^2}{2} - \frac{y^2}{2} + V(\xi, \phi)} - \frac{\partial V}{\partial \xi}. \end{cases} \quad (67)$$

Notice that the reflection

$$\begin{aligned} x &\rightarrow -x, \\ y &\rightarrow -y, \\ t &\rightarrow -t \end{aligned} \quad (68)$$

transforms the system into one describing the cosmological evolution with the opposite sign of the Hubble parameter. The stationary points of the system (67) are given by

$$x = 0, \quad y = 0, \quad \frac{\partial V}{\partial \phi} = 0, \quad \frac{\partial V}{\partial \xi} = 0. \quad (69)$$

3.1. Model I

In this subsection we shall analyze the cosmological model with two fields - standard scalar and phantom, described by the potential (65). For this potential the system of equations (67) reads

$$\begin{cases} \dot{\phi} = x, \\ \dot{\xi} = y, \\ \dot{x} = -3\text{sign}(h)x\sqrt{\frac{x^2}{2} - \frac{y^2}{2} + \frac{2}{9t_R^2}\sinh^2(3\xi/4)\cosh^6(3\xi/4)e^{-3\sqrt{2}\phi}} \\ \quad + \frac{2\sqrt{2}}{3t_R^2}\sinh^2(3\xi/4)\cosh^6(3\xi/4)e^{-3\sqrt{2}\phi}, \\ \dot{y} = -3\text{sign}(h)y\sqrt{\frac{x^2}{2} - \frac{y^2}{2} + \frac{2}{9t_R^2}\sinh^2(3\xi/4)\cosh^6(3\xi/4)e^{-3\sqrt{2}\phi}} \\ \quad + \frac{\sinh(3\xi/4)\cosh^5(3\xi/4)}{3t_R^2}\left[\frac{1}{3}\cosh^2(3\xi/4) + \sinh^2(3\xi/4)\right]e^{-3\sqrt{2}\phi}. \end{cases} \quad (70)$$

It is easy to see that there are stationary points

$$\phi = \phi_0, \quad \xi = 0, \quad x = 0, \quad y = 0, \quad (71)$$

where ϕ_0 is arbitrary. For these points the potential and hence the Hubble variable vanish. Thus, we have a static cosmological solution. We should study the behavior of our system in the neighborhood of the point (71) in linear approximation:

$$\begin{cases} \dot{\phi} = x, \\ \dot{\xi} = y, \\ \dot{x} = 0 \\ \dot{y} = +\frac{\xi}{4t_R^2}e^{-3\sqrt{2}\phi_0}. \end{cases} \quad (72)$$

One sees that the dynamics of ϕ in this approximation is frozen and hence we can focus on the study of the dynamics of the variables ξ, y . The eigenvalues of the corresponding subsystem of two equations are

$$\lambda_{1,2} = \mp \frac{e^{-3\phi_0/\sqrt{2}}}{2t_R}. \quad (73)$$

These eigenvalues are real and have opposite signs, so one has a saddle point in the plane (ξ, y) and this means that the points (71) are unstable.

One can make another qualitative observation. Freezing the dynamics of ξ independently of ϕ , namely choosing $y = 0, \xi = 0$, which implies also $\dot{y} = \ddot{\xi} = 0$, one has the following equation of motion for ϕ :

$$\ddot{\phi} + 3h\phi = 0. \quad (74)$$

Equation (74) is nothing but the Klein-Gordon equation for a massless scalar field on the Friedmann background, whose solution is

$$\phi(t) = \frac{\sqrt{2}}{3} \ln t, \quad (75)$$

and which gives a Hubble variable

$$h(t) = \frac{1}{3t}. \quad (76)$$

This is an evolution of the flat Friedmann universe, filled with stiff matter with the equation of state $p = \varepsilon$. It describes a universe, born from the Big Bang singularity and infinitely expanding. Naturally, for the opposite sign of the Hubble parameter, one has the contracting universe ending in the Big Crunch cosmological singularity.

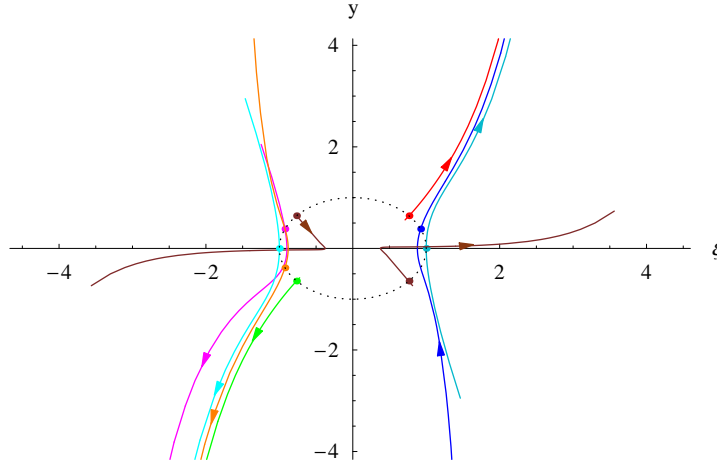


Figure 1. An example of section of the 4D phase space obtained with numerical calculations. We can see a series of trajectories corresponding to different choices of the initial conditions. Every initial condition in the graphic is chosen on the dashed “ellipse” centered in the origin and is emphasized by a colored dot. A “saddle point-like” structure of the set of the trajectories clearly emerges.

Now, we describe some results of numerical calculations to have an idea about the structure of the set of possible cosmological evolutions coexisting in the model under consideration. We have carried out two kinds of simulations. First, we have considered neighborhood of the plane y, ξ with the initial conditions on the field ϕ such that $\phi(0) = 0, \dot{\phi}(0) = 0$ (see Figure 1). The initial conditions for the phantom field were chosen in such a way that the sum of absolute values of the kinetic and potential energies were fixed. Then, running the time back and forward we have seen that the absolute majority of the cosmological evolutions began at the singularity of the “anti-Big Rip” type (Figure 2). Namely, the initial cosmological singularity were characterized by an infinite value of the cosmological radius and an infinite negative value of its time derivative (and also of the Hubble variable). Then the universe squeezes, being dominated by the phantom scalar field ξ . At some moment the universe passes the phantom divide line and the universe continues squeezing but with $\dot{h} < 0$. Then it achieves the minimal value of the cosmological radius and an expansion begins. At some moment the universe undergoes the second phantom divide line crossing and its expansion becomes super-accelerated culminating in an encounter with a Big Rip singularity. Apparently this scenario is very different from the standard cosmological scenario and with its phantom version Bang-to-Rip, which has played a role of an input in the construction of our potentials. The second procedure, which we have used is the consideration of trajectories close to our initial trajectory of the Bang-to-Rip

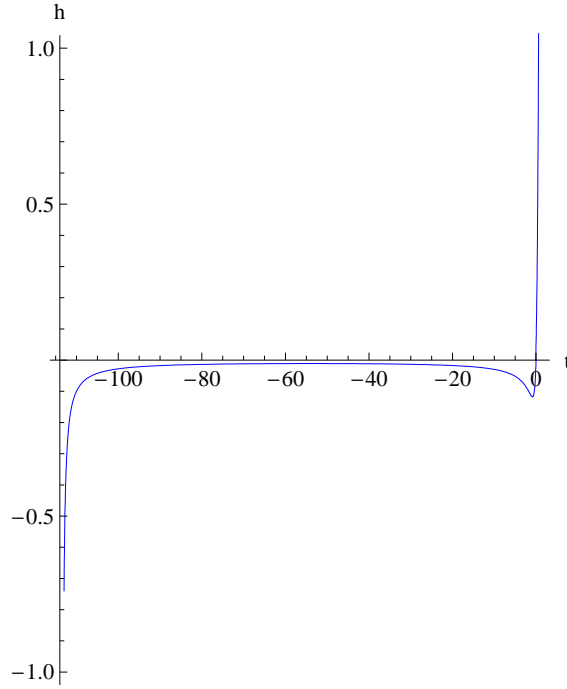


Figure 2. Typical behavior of the Hubble parameter for the model I trajectories. The presence of two stationary points (namely a maximum followed by a minimum) indicates the double crossing of the phantom divide line. Both the initial and final singularities are characterized by a Big Rip behavior, the first is a contraction and the second is an expansion.

type. The numerical analysis shows that this trajectory is unstable and the neighboring trajectories again have anti-Big Rip - double crossing of the phantom line - Big Rip behavior described above. However, it is necessary to emphasize that a small subset of the trajectories of the Bang-to-Rip type exist, being not in the vicinity of our initial trajectory.

3.2. Model II

In this subsection we shall study the cosmological model with the potential (58). Now the system of equations (67) looks like

$$\begin{cases} \dot{\phi} = x, \\ \dot{\xi} = y, \\ \dot{x} = -3\text{sign}(h)x\sqrt{\frac{x^2}{2} - \frac{y^2}{2} + \frac{2}{9t_R^2}\cosh^6(3\phi/4)\exp\{3\sqrt{2}\xi\}} \\ \quad - \frac{1}{t_R^2}\cosh^5(3\phi/4)\sinh(3\phi/4)e^{3\sqrt{2}\xi}, \\ \dot{y} = -3\text{sign}(h)y\sqrt{\frac{x^2}{2} - \frac{y^2}{2} + \frac{2}{9t_R^2}\cosh^6(3\phi/4)\exp\{3\sqrt{2}\xi\}} \\ \quad + \frac{2\sqrt{2}}{3t_R^2}\cosh^6(3\phi/4)e^{3\sqrt{2}\xi}. \end{cases} \quad (77)$$

Notice that the potential (58) has an additional reflection symmetry

$$V_2(\xi, \phi) = V_2(\xi, -\phi). \quad (78)$$

This provides the symmetry with respect to the origin in the plane (ϕ, x) . The system (77) has no stationary points. However, there is an interesting point

$$\phi = 0, \quad x = 0, \quad (79)$$

which freezes the dynamics of ϕ and hence, permits to consider independently the dynamics of ξ and y , described by the subsystem

$$\begin{cases} \dot{\xi} = y, \\ \dot{y} = -3\text{sign}(h)y\sqrt{-\frac{y^2}{2} + \frac{2}{9t_R^2}e^{3\sqrt{2}\xi}} + \frac{2\sqrt{2}}{3t_R^2}e^{3\sqrt{2}\xi}. \end{cases} \quad (80)$$

Apparently, the evolution of the universe is driven now by the phantom field and is subject to superacceleration.

In this case the qualitative analysis of the differential equations for ξ and y , confirmed by the numerical simulations gives a predictable result: being determined by the only phantom scalar field the cosmological evolution is characterized by the growing positive value of h . Namely, the universe begins its evolution from the anti-Big Rip singularity ($h = -\infty$) then h is growing passing at some moment of time the value $h = 0$ (the point of minimal contraction of the cosmological radius $a(t)$) and then expands ending its evolution in the Big Rip cosmological singularity ($h = +\infty$).

Another numerical simulation can be done by fixing initial conditions for the phantom field as $\xi(0) = 0, y(0) = 0$ (see Figure 3). Choosing various values of the initial conditions for the scalar field $\phi(0), x(0)$ around the point of freezing $\phi = 0, x = 0$ we found two types of cosmological trajectories:

1. The trajectories starting from the anti-Big Rip singularity and ending in the Big Rip after the double crossing of the phantom divide line. These trajectories are similar to those discussed in the preceding subsection for the model I.
2. The evolutions of the type Bang-to-Rip.

Then we have carried out the numerical simulations of cosmological evolutions, choosing the initial conditions around the point of the phantomization point with the coordinates

$$\begin{aligned} \phi(0) &= 0, \\ x(0) &= \frac{\sqrt{2}}{3}, \\ \xi(0) &= \frac{4}{3}\text{arctanh}\frac{1}{\sqrt{2}}, \\ y(0) &= \frac{\sqrt{2}}{3}. \end{aligned} \quad (81)$$

This analysis shows that in contrast to the model I, here the standard phantomization trajectory is stable and the trajectories of the type Bang-to-Rip are not exceptional, though less probable than those of the type anti-Big Rip to Big Rip.

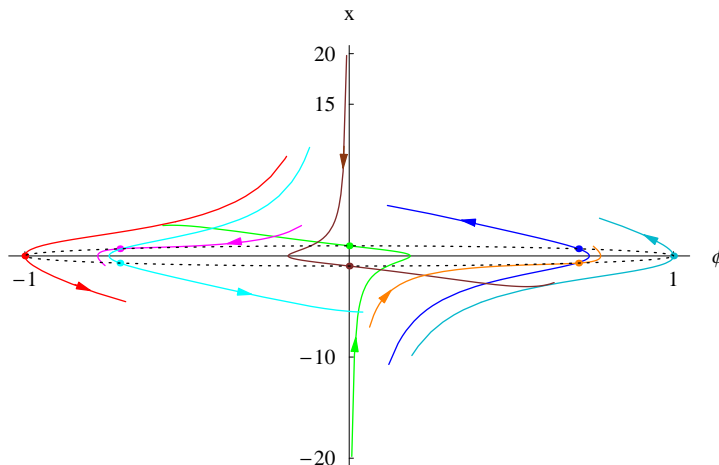


Figure 3. An example of section of the 4D phase space for the model II obtained with numerical calculations. We can see a family of different trajectories corresponding to different choices of the initial conditions. Every initial condition in the graphic is chosen on the dashed “ellipse”. The origin represents the point of freezing.

4. Conclusion

We have considered the problem of reconstruction of the potential in a theory with two scalar fields (one standard and one phantom) starting with a given cosmological evolution. It is known (see e.g. [15, 16, 17]) that in the case of the only scalar field this potential is determined uniquely as well as the initial conditions for the scalar field, reproducing the given cosmological evolution. Changing the initial conditions, one can find a variety of cosmological evolutions, sometimes qualitatively different from the “input” one (see e.g. [17]). In the case of two fields the procedure of reconstruction becomes much more involved. As we have shown here, there is a huge variety of different potentials reproducing the given cosmological evolution (a very simple one in the case, which we have explicitly studied here). Every potential entails different cosmological evolutions, depending on the initial conditions.

It is interesting that the existence of different dynamics of scalar fields corresponding to the same evolution of the Hubble parameter $h(t)$ can imply some observable consequences connected with the possible interactions of the scalar fields with other fields. Indeed, in this case, the time dependence of the scalar fields considered above can directly affect physically observable quantities. We are going to consider this topic elsewhere [25].

Acknowledgments

We are grateful to G. Venturi for fruitful discussions and to M. Szydlowski, S.Yu. Vernov and A.I. Zhuk for useful correspondence. This work was partially supported by Grants RFBR 08-02-00122-a, 05-02-17450 and LSS-3251.2008.2, LSS-1757.2006.2. The work of A.A. was also supported by Grant FPA2007-66665, 2007PIV10046 and Program RNP

2.1.1.1112.

References

- [1] Riess A et al. 1998 *Astron. J.* **116** 1009; Perlmutter S J et al. 1999 *Astroph. J.* **517**, 565.
- [2] Sahni V and Starobinsky A A 2000 *Int. J. Mod. Phys. D* **9** 373; Padmanabhan T 2003 *Phys. Rep.* **380** 235; Peebles P J E and Ratra B 2003 *Rev. Mod. Phys.* **75** 559; Sahni V 2002 *Class. Quantum Grav.* **19** 3435; Copeland E J, Sami M and Tsujikawa S 2006 *Int. J. Mod. Phys. D* **15** 1753; Sahni V and Starobinsky A A 2006 *Int. J. Mod. Phys. D* **15** 2105.
- [3] Caldwell R R, Dave R and Steinhardt P J 1998 *Phys. Rev. Lett.* **80** 1582; Frieman J A, Waga I 1998 *Phys. Rev. D* **57** 4642.
- [4] Armendariz-Picon C, Damour T and Mukhanov V 1999 *Phys. Lett. B* **458** 209; Armendariz-Picon C, Mukhanov V and Steinhardt P J 2000 *Phys. Rev. Lett.* **85** 4438; Armendariz-Picon C, Mukhanov V, Steinhardt P J 2001 *Phys. Rev. D* **63** 103510; Chiba T, Okabe T, Yamaguchi M 2000 *Phys. Rev. D* **62** 023511; Kamenshchik A, Moschella U and Pasquier V 2001 *Phys. Lett. B* **511** 265; Bilic N, Tupper G B, and Violeer R 2002 *Phys. Lett. B* **535** 17; Fabris J C, Gonsalves V B and de Souza P E 2002 *Gen. Rel. Grav.* **34** 53; Bento M C, Bertolami O and Sen A A 2002 *Phys. Rev. D* **66** 043507; Gorini V, Kamenshchik A and Moschella U 2003 *Phys. Rev. D* **67** 063509; Sen A 2002 *JHEP* **0204** 048; Gibbons G W 2002 *Phys. Lett. B* **537** 1; Fairbairn M and Tytgat M H G 2002 *Phys. Lett. B* **546** 1; Mukohyama S 2002 *Phys. Rev. D* **66** 024009; Padmanabhan T and Choudhury T R 2002 *Phys. Rev. D* **66** 081301(R); Sami M, Chingangbam P and Qureshi T 2002 *Phys. Rev. D* **66** 043530; Felder G N, Kofman L and Starobinsky A A 2002 *JHEP* **0209** 026; Frolov A, Kofman L and Starobinsky A 2002 *Phys. Lett. B* **545** 8; Padmanabhan T 2002 *Phys. Rev. D* **66** 021301(R); Feinstein A 2002 *Phys. Rev. D* **66** 063511; Abramo L W and Finelli F 2003 *Phys. Lett. B* **575** 165; Bagla J S, Jassal H K and Padmanabhan T 2003 *Phys. Rev. D* **67** 063504; Gibbons G 2003 *Class. Quant. Grav.* **20** S321; Causse M B A rolling tachyon field for both dark energy and dark halos of galaxies *Preprint astro-ph/0312206*; Paul B C and Sami M 2004 *Phys. Rev. D* **70** 027301; Sen A 2005 *Phys. Scripta* **T 117** 70; Garousi M R, Sami M and Tsujikawa S 2004 *Phys. Rev. D* **70** 043536; Calcagni G 2004 *Phys. Rev. D* **69** 103508; Aguirregabiria J M and Lazkoz R 2004 *Mod. Phys. Lett. A* **19** 927; Aguirregabiria J M and Lazkoz R 2004 *Phys. Rev. D* **69** 123502; Herrera R, Pavon D and Zimdahl W 2004 *Gen. Rel. Grav.* **36** 2161; Barnaby N 2004 *JHEP* **0407** 025; Allemandi G, Borowiec A and Francaviglia M 2004 *Phys. Rev. D* **70** 043524; Zimdahl W Interacting Dark Energy and Cosmological Equations of State *Preprint gr-qc/0505056*; Allemandi G, Borowiec A, Francaviglia M and Odintsov S D 2005 *Phys. Rev. D* **72** 063505.
- [5] Cooper F and Venturi G 1981 *Phys. Rev. D* **24** 3338; Ratra B and Peebles P J E 1988 *Phys. Rev. D* **37** 3406; Peebles P J E and Ratra B 1988 *Astrophys. J. Lett.* **325** L17; Wetterich C 1988 *Nucl. Phys. B* **302** 668.
- [6] Caldwell R R 2002 *Phys. Lett. B* **545** 23; Gonzalez-Diaz P F *Phys. Lett. B* **586** 1; Singh P, Sami M and Dadhich N *Phys. Rev. D* **68** 023522; Nojiri S and Odintsov S D *Phys. Lett. B* **562** 147; Capozziello S, Nojiri S and Odintsov S D *Phys. Lett. B* **632** 597; Johri V B 2004 *Phys. Rev. D* **70** 041303(R); Onemli V K and Woodard R P 2000 *Class. Quant. Grav.* **19** 4607; Hannestad S and Mortsell E 2002 *Phys. Rev. D* **66** 063508; Carroll S M, Hoffman M and Trodden M *Phys. Rev. D* **68** 023509; Frampton P H Stability Issues for $w < -1$ Dark Energy *Preprint hep-th/0302007*; Elizalde E, Nojiri S and Odintsov S D *Phys. Rev. D* **70** 043539; Gonzalez-Diaz P F and Siguenza C L 2004 *Nucl. Phys. B* **697** 363; Gibbons G W Phantom Matter and the Cosmological Constant *Preprint hep-th/0302199*; McInnes B What If $w < -1$? *Preprint astro-ph/0210321*; Chimento L P and Lazkoz R 2003 *Phys. Rev. Lett.* **91** 211301; Dabrowski M P, Stachowiak T and Szydlowski M 2003 *Phys. Rev. D* **68** 103519; Singh P, Sami M and Dadhich N 2003 *Phys. Rev. D* **68** 023522; Gonzalez-Diaz P F 2004 *Phys. Rev. D* **69** 063522; Onemli V K and Woodard R P 2004 *Phys. Rev. D* **70** 107301; Sami M and Toporensky A 2004 *Mod. Phys. Lett. A* **19** 1509; Stefancic H

- 2004 *Eur. Phys. J. C* **36** 523; Stefancic H 2004 *Phys. Lett. B* **586** 5; Brunier T, Onemli V K and Woodard R P 2005 *Class. Quantum Grav.* **22** 59; Santos J and Alcaniz J S 2005 *Phys. Lett. B* **619** 11; Carvalho F C and Saa A 2004 *Phys. Rev. D* **70** 087302; Melchiorri A, Mersini L, Odman C J and Trodden M 2003 *Phys. Rev. D* **68** 043509; Cline J M, Jeon S and Moore J D 2004 *Phys. Rev. D* **70** 043543; Guo Z K, Piao Y S, Zhang X M and Zhang Y Z 2005 *Phys. Lett. B* **608** 177; Feng B, Wang X L and Zhang X M 2005 *Phys. Lett. B* **607** 35; Aref'eva I Y, Koshelev A S and Vernov S Y Exactly Solvable SFT Inspired Phantom Model to be published in *Theor. Math. Phys. Preprint* astro-ph/0412619; Zhang X F, Li H, Piao Y S and Zhang X M Two-field Models of Dark Energy with Equation of State Across -1 *Preprint* astro-ph/0501652; Calcagni G 2005 *Phys. Rev. D* **71** 023511; Li M, Feng B and Zhang X 2005 *JCAP* **0512** 002; Babichev E, Dokuchaev V and Eroshenko Yu 2005 *Class. Quant. Grav.* **22** 143; Anisimov A, Babichev E and Vikman A 2005 *JCAP* **0506** 006; Nojiri S, Odintsov S D and Tsujikawa S 2005 *Phys. Rev. D* **71** 063004; Gumjudpai B, Naskar T, Sami M and Tsujikawa S 2005 *JCAP* **0506** 007; Sami M, Toporensky A, Tretjakov P V and Tsujikawa S 2005 *Phys. Lett. B* **619** 193; Nojiri S and Odintsov S D 2006 *Gen. Relativ. Grav.* **38** 1285; Brevik I 2006 *Gen. Relativ. Grav.* **38** 1317.
- [7] Alam U, Sahni V, Saini T D and Starobinsky A A 2004 *Mon. Not. Roy. Astron. Soc.* **354** 275; Padmanabhan T and Choudhury T R 2003 *Mon. Not. Roy. Astron. Soc.* **344** 823; Choudhury T R and Padmanabhan T 2005 *Astron. Astrophys.* **429** 807; Wang Y and Mukherjee P 2004 *Astrophys. J.* **606** 654; Huterer D and Cooray A 2005 *Phys. Rev. D* **71** 023506; Daly R A and Djorgovski S G 2003 *Astrophys. J.* **597** 9; Alcaniz J S 2004 *Phys. Rev. D* **69** 083521; Lima J A S, Cunha J V and Alcaniz J S 2003 *Phys. Rev. D* **68** 023510; Alam U, Sahni V and Starobinsky A A 2004 *JCAP* **0406** 008; Wang Y and Freese K 2006 *Phys. Lett. B* **632** 449; Upadhye A, Ishak M and Steinhardt P J 2005 *Phys. Rev. D* **72** 063501; Dicus D A and Repko W W 2004 *Phys. Rev. D* **70** 083527; Espana-Bonet C and Ruiz-Lapuente P Dark Energy as an Inverse Problem *Preprint* hep-ph/0503210; Wei Y H The power-law expansion universe and dark energy evolution *Preprint* astro-ph/0405368; Jassal H K, Bagla J S, and Padmanabhan T 2005 *Mon. Not. Roy. Astron. Soc. Letters* **356** L11; Nesseris S and Perivolaropoulos L 2004 *Phys. Rev. D* **70** 043531; Lazkoz R, Nesseris S and Perivolaropoulos L 2005 *JCAP* **0511** 010; Jassal H K, Bagla J S and Padmanabhan T 2005 *Phys. Rev. D* **72** 103503; Feng B, Li M, Piao Y S and Zhang X 2006 *Phys. Lett. B* **634** 101-105; Xia J Q, Feng B and Zhang X M 2005 *Mod. Phys. Lett. A* **20** 2409.
- [8] Cabre A, Gaztanaga E, Manera M, Fosalba P and Castander F 2006 *Mon. Not. Roy. Astron. Soc. Lett.* **372** L23-L27.
- [9] Barger V, Gao Y and Marfatia D 2007 *Phys. Lett. B* **648** 127; Gong A and Wang A 2007 *Phys. Rev. D* **75** 0435520; Alam U, Sahni V and Starobinsky A A 2007 *JCAP* **0702** 011; Nesseris S and Perivolaropoulos L 2007 *JCAP* **0702** 025; Zhao G B, Xia J Q, Li H et al. Probing for dynamics of dark energy and curvature of universe with latest cosmological observations *Preprint* astro-ph/0612728; Serra P, Heavens A and Melchiorri A 2007 *MNRAS* **379** 1,169; Davis T M, Mortsell E, Sollerman J et al. Scrutinizing Exotic Cosmological Models Using ESSENCE Supernova Data Combined with Other Cosmological Probes *Preprint* astro-ph/0701510; Wright E L Constraints on Dark Energy from Supernovae, Gamma Ray Bursts, Acoustic Oscillations, Nucleosynthesis and Large Scale Structure and the Hubble constant *Preprint* astro-ph/0701584; Wang Y and Mukherjee P Observational Constraints on Dark Energy and Cosmic Curvature *Preprint* astro-ph/0703780.
- [10] Boisseau B, Esposito-Farese G, Polarski D and Starobinsky A A 2000 *Phys. Rev. Lett.* **85** 2236; Esposito-Farese G and Polarski D 2001 *Phys. Rev. D* **63** 063504; Vikman A 2005 *Phys. Rev. D* **71** 023515; Perivolaropoulos L 2005 *Phys. Rev. D* **71** 063503; McInnes B 2005 *Nucl. Phys. B* **718** 55; Aref'eva I Ya, Koshelev A S and Vernov S Yu 2005 *Phys. Rev. D* **72** 064017; Perivolaropoulos L 2005 *JCAP* **0510** 001; Caldwell R R and Doran M 2005 *Phys. Rev. D* **72** 043527; Sahni V and Shtanov Yu 2003 *JCAP* **0311** 014; Sahni V and Wang L 2000 *Phys. Rev. D* **62** 103517; Wei H, Cai R G and Zeng D F 2005 *Class. Quant. Grav.* **22** 3189; Wei H, Cai R G 2005 *Phys. Rev. D*

- 72** 123507; Wei H, Cai R G 2006 *Phys. Lett. B* **634** 9; Cai R G and Wang A 2005 *JCAP* **0503** 002; Gannouji R, Polarski D, Ranquet A and Starobinsky A A 2006 *JCAP* **0609** 016.
- [11] Andrianov A A, Cannata F and Kamenshchik A Yu 2005 *Phys. Rev. D* **72** 043531.
- [12] Cannata F and Kamenshchik A Yu Networks of cosmological histories, crossing of the phantom divide line and potentials with cusps (to appear in *Int. J. Mod. Phys. D*) *Preprint* gr-qc/0603129.
- [13] Andrianov A A, Cannata F and Kamenshchik A Yu 2006 *Int. J. Mod. Phys. D* **15** 1299; Andrianov A A, Cannata F and Kamenshchik A Yu 2006 *J. Phys. A* **39** 9975.
- [14] Feng B, Wang X and Zhang X 2005 *Phys. Lett. B* **607** 35; Guo Z K, Piao Y S, Zhang X and Zhang Y Z 2005 *Phys. Lett. B* **608** 177; Hu W 2005 *Phys. Rev. D* **71** 047301; Perivolaropoulos L 2005 *Phys. Rev. D* **71** 063503; Caldwell R R and Doran M 2005 *Phys. Rev. D* **72** 043527.
- [15] Starobinsky A A 1998 *JETP Lett.* **68** 757.
- [16] Burd A B and Barrow J D 1988 *Nucl. Phys. B* **308** 929; Barrow J D 1990 *Phys. Lett. B* **235** 40.
- [17] Gorini V, Kamenshchik A, Moschella U and Pasquier V 2004 *Phys. Rev. D* **69** 123512.
- [18] Zhuravlev V M, Chervon S V and Shchigolev V K 1998 *JETP* **87** 223; Chervon S V and Zhuravlev V M The cosmological model with an analytic exit from inflation *Preprint* gr-qc/9907051; Yurov A V Phantom scalar fields result in inflation rather than Big Rip *Preprint* astro-ph/0305019; Yurov A V and Vereshchagin S D 2004 *Theor. Math. Phys.* **139** 787.
- [19] Guo Z K, Ohta N and Zhang Y Z 2007 *Mod. Phys. Lett. A* **22** 883; Guo Z K, Ohta N and Zhang Y Z 2005 *Phys. Rev. D* **72** 023504.
- [20] Zhuk A 1996 *Class. Quant. Grav.* **13** 2163.
- [21] Szydlowski M and Czaja W 2004 *Phys. Rev. D* **69** 083507; Szydlowski M and Czaja W 2004 *Phys. Rev. D* **69** 083518; Szydlowski M 2005 *Int. J. Mod. Phys. A* **20** 2443; Szydlowski M, Hrycyna O and Krawiec A 2007 *JCAP* **0706** 010.
- [22] Vernov S Yu Construction of Exact Solutions in Two-Fields Models and the Crossing of the Cosmological Constant Barrier *Preprint* astro-ph/0612487.
- [23] Regoli D 2007 *Mg. Thesis* University of Bologna (in Italian).
- [24] Belinsky V A, Khalatnikov I M, Grishchuk L P and Zeldovich Ya B 1985 *Sov. Phys. JETP* **62** 195; Belinsky V A, Khalatnikov I M, Grishchuk L P and Zeldovich Ya B 1985 *Phys. Lett. B* **155** 232.
- [25] Andrianov A A, Cannata F, Kamenshchik A Yu and Regoli D, work in progress.